RECITATION 5

DERIVATIVES, BUT NOW THEY'RE EASY

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Section 1. *Properties of Differentiation*

We've encountered derivatives so far, but let's investigate differentiation, meaning the map between (differentiable) functions and their derivatives. By knowing more about differentiation, we can break down complicated functions into simpler parts, and from there evaluate the derivatives of the simpler parts. In particular, since limits work well with addition, we have

1 • 1. Result

Let f and g be two functions differentiable at x. Therefore $(f+g)'(x) = f'(x) + g'(x)$.

Proof .:.

For f and g to be differentiable at x, we must have convergence of the following limits to finite numbers:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{and} \quad \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}.$$

By the definition of the derivative, and since both converge to finite numbers,

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right) \\ &= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h}\right) + \lim_{h\to 0} \left(\frac{g(x+h)-g(x)}{h}\right) \\ &= f'(x) + g'(x) \dashv\end{aligned}$$

This is the simplest property of differentiation to prove: it distributes over addtion $\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}}{\mathrm{d}x} f + \frac{\mathrm{d}}{\mathrm{d}x} g$. We also have more complicated properties with more complicated proofs. But the nice thing about these properties is that they allow us to break down almost any function that was built up with sines, cosines, exponentials, logarithms, polynomials, and of course multiplication, division, and addition.

1 • 2. Result (Differentiation Rules)

Let f and g be two functions differentiable at x, and let c be a constant. Therefore,

- (scalar multiplication) $(c \cdot f)'(x) = c \cdot f'(x)$.
- (The product rule) $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.
- (The chain rule) $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$, where $f \circ g$ is the function defined by $f \circ g(x) = f(g(x))$.
- (The power rule) $\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}$ for any n (e.g. $n = 1/2$, or $n = \pi$, or $n = 4$, or $n = \sqrt{5}$, and so on).

From this, we get the so-called "quotient rule". Honestly, this rule doesn't need to exist, but many sources list is as important, and not just a redundancy. In many cases, using the quotient rule can make things less simplified. Regardless, it's a part of the curriculum.

1 • 3. Corollary (The Quotient Rule)

Let f and g be two functions differentiable at x such that $g(x) \neq 0$. Therefore
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Proof .:.

Note that $f/g = f \cdot \frac{1}{g}$. By the product rule from Differentiation Rules (1 • 2),
$$\left(\frac{f}{g}\right)'(x) = f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)}.$$
By the power rule, $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} = \frac{\mathrm{d}}{\mathrm{d}x}x^{-1} = -x^{-2} = \frac{-1}{x^2}$. So by the chain rule from Differentiation Rules (1 • 2), $\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{g(x)} = \frac{-1}{(g(x))^2}g'(x)$. Hence we have
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)}{g(x)} + \frac{f(x)\cdot(-g'(x))}{(g(x))^2} = \frac{f'(x)g(x) - f(x)\cdot g'(x)}{(g(x))^2} \dashv$$

In addition to those rules, we also have some hard to evaluate derivatives.

1 • 4. Result (Special Derivatives)

- (Differentiation of exponentials) $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$.
- (Differentiation of logarithms) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$.
- (Differentiation of trigonometric functions) $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$, and $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$ (note the minus sign).

The proofs of these derivatives can be found in the book, and mostly revolve around whatever formal definition we choose for things like e and the trigonometric functions. Ultimately, they are not especially interesting, but they are widely used. Importantly, this allows us to evaluate the derivative of other, similar functions.

1 • 5. Corollary

- $\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2(x) = \frac{1}{\cos^2(x)}$.
- $\frac{\mathrm{d}}{\mathrm{d}x}2^x = \ln(2)2^x$. In general, $A^x = \ln(A)\cdot A^x$ for any $A > 0$.
- $\frac{\mathrm{d}}{\mathrm{d}x}6x = 6$. In general, $\frac{\mathrm{d}}{\mathrm{d}x}Ax = A$ for any A, matching up with the idea of the derivative being a slope.

Proof .:.

Note that $\tan x = \sin(x)/\cos(x)$ so that by The Quotient Rule (1 • 3) or Differentiation Rules (1 • 2),
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \cos(x)\cdot\frac{1}{\cos(x)} - \sin(x)\cdot\frac{1}{\cos^2(x)}\cdot(-\sin x) = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$
An easier derivative, note that $2^x = e^{\ln(2)\cdot x}$ So by the chain rule and Special Derivatives (1 • 4),
$$\frac{\mathrm{d}}{\mathrm{d}x}e^{\ln(2)\cdot x} = e^{\ln(2)\cdot x}\cdot\ln(2) = \ln(2)\cdot 2^x.$$
And the easiest of the three, note that as a constant, $\frac{\mathrm{d}}{\mathrm{d}x}6x = 6\cdot\frac{\mathrm{d}}{\mathrm{d}x}x = 6\cdot 1\cdot x^0 = 6\cdot 1\cdot 1 = 6.$ $\dashv$

Ultimately, using Differentiation Rules (1 • 2) and Special Derivatives (1 • 4) allows one to *mechanically*, without thinking, evaluate just about every derivative that will be thrown at you, just by applying the rules according to how the defined function was built up.

Section 2. Exercises

Exercise 1

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\cot(x)$.

Solution .:.

$\cot(x) = \frac{1}{\tan(x)}$. We know $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \frac{1}{\cos^2(x)}$ by Corollary 1 • 5 so that by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot(x) = \frac{-1}{\tan^2(x)} \cdot \frac{1}{\cos^2(x)} = \frac{-\cos^2(x)}{\sin^2(x)} \cdot \frac{1}{\cos^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x).$$

Exercise 2

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 5x + 6)$ using the product rule and using the power rule.

Solution .:.

Using the power rule, the derivative is $2x - 5 \cdot x^0 = 2x - 5$. Using the product rule, note that $x^2 - 5x + 6 = (x-3)(x-2)$ so that

$$\frac{\mathrm{d}}{\mathrm{d}x}(x-3)(x-2) = 1 \cdot (x-2) + (x-3) \cdot 1 = 2x - 5.$$

Exercise 3

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\sin(e^x)$.

Solution .:.

Using the chain rule and Special Derivatives (1 • 4), the derivative is $\cos(e^x) \cdot e^x$.

Exercise 4

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}5^x$.

Solution .:.

Note that $5^x = e^{\ln(5)\cdot x}$ so differentiation yields $\ln(5) \cdot e^{\ln(5)\cdot x} = \ln(5) \cdot 5^x$.

Exercise 5

For what x does $f'(x) = 0$? Here, $f(x) = x^3 - 6x^2 - 15x + 9$.

Solution .:.

Note that $f'(x) = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x-5)(x+1)$. Hence $f'(x) = 0$ iff $x = 5$ or $x = -1$. ⊣

Exercise 6

Write $f(x) = \ln(\sin x)$. Compute $f'(x)$.

Solution .:.

By the chain rule, and Special Derivatives (1 • 4), $f'(x) = \frac{1}{\sin x} \cdot \cos(x) = \cot x$.

Exercise 7

Write $f(x) = \ln(e^x)$. Using the chain rule, show $f'(x) = 1$ (obviously $f(x) = x$ implies this).

Proof .:.

$\frac{\mathrm{d}}{\mathrm{d}x}\ln(x) = \frac{1}{x}$. $\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x$ so that $f'(x) = \frac{1}{e^x} \cdot e^x = 1$.